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Qualifications

PEARSON EDEXCEL INTERNATIONAL GCSE (9–1)

MATHEMATICS A

Student Book 2

David Turner, Ian Potts

eBook included

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EDEXCEL INTERNATIONAL GCSE (9–1)

MATHEMATICS A

Student Book 2

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Ian Potts

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Dedicated to Viv Hony who started the whole project.

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UNIT 6

UNIT 7

UNIT 8

UNIT 9

UNIT 10

ABOUT THIS BOOK

This two-book series is written for students following the Edexcel International GCSE (9–1) Maths A Higher Tier specification. There is a Student Book for each year of the course.

The course has been structured so that these two books can be used in order, both in the classroom and for independent learning.

Each book contains five units of work. Each unit contains five sections in the topic areas: *Number, Algebra, Sequences, Graphs*, *Shape and Space, Sets* and *Handling Data*.

Each unit contains concise explanations and worked examples, plus numerous exercises that will help you build up confidence.

Non-starred and starred parallel exercises are provided, to bring together basic principles before being challenged with more difficult questions. These are supported by parallel revision exercises at the end of each chapter.

Challenges, which provide questions applying the basic principles in unusual situations, feature at the back of the book along with *Fact Finders* which allow you to practise comprehension of real data.

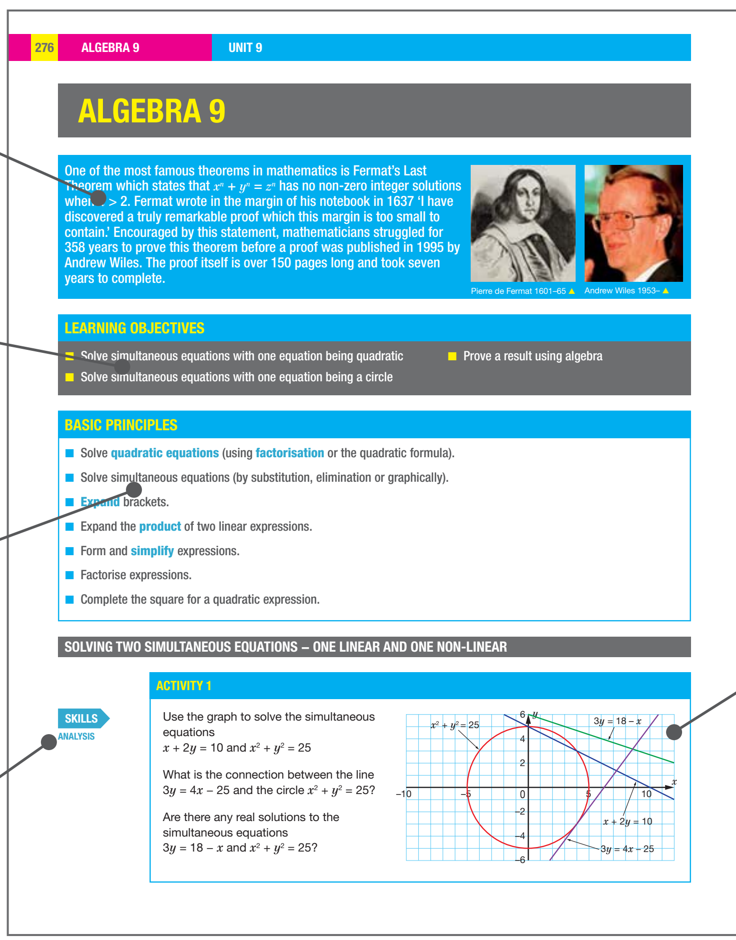
Points of Interest put the maths you are about to learn in a real-world context.

Learning Objectives show what you will learn in each lesson.

Basic Principles outline assumed knowledge and key concepts from the beginning.

Transferable Skills are highlighted to show what skill you are using and where.

Activities are a gentle way of introducing a topic.


276 ALGEBRA 9 UNIT 9

ALGEBRA 9

One of the most famous theorems in mathematics is Fermat's Last Theorem which states that $x^n + y^n = z^n$ has no non-zero integer solutions when $n > 2$. Fermat wrote in the margin of his notebook in 1637 'I have discovered a truly remarkable proof which this margin is too small to contain.' Encouraged by this statement, mathematicians struggled for 358 years to prove this theorem before a proof was published in 1995 by Andrew Wiles. The proof itself is over 150 pages long and took seven years to complete.

Pierre de Fermat 1601–65 ▲ Andrew Wiles 1953– ▲

LEARNING OBJECTIVES

- Solve simultaneous equations with one equation being quadratic
- Solve simultaneous equations with one equation being a circle
- Prove a result using algebra

BASIC PRINCIPLES

- Solve quadratic equations (using factorisation or the quadratic formula).
- Solve simultaneous equations (by substitution, elimination or graphically).
- Expand brackets.
- Expand the product of two linear expressions.
- Form and simplify expressions.
- Factorise expressions.
- Complete the square for a quadratic expression.

SOLVING TWO SIMULTANEOUS EQUATIONS – ONE LINEAR AND ONE NON-LINEAR

SKILLS ANALYSIS

ACTIVITY 1

Use the graph to solve the simultaneous equations $x + 2y = 10$ and $x^2 + y^2 = 25$

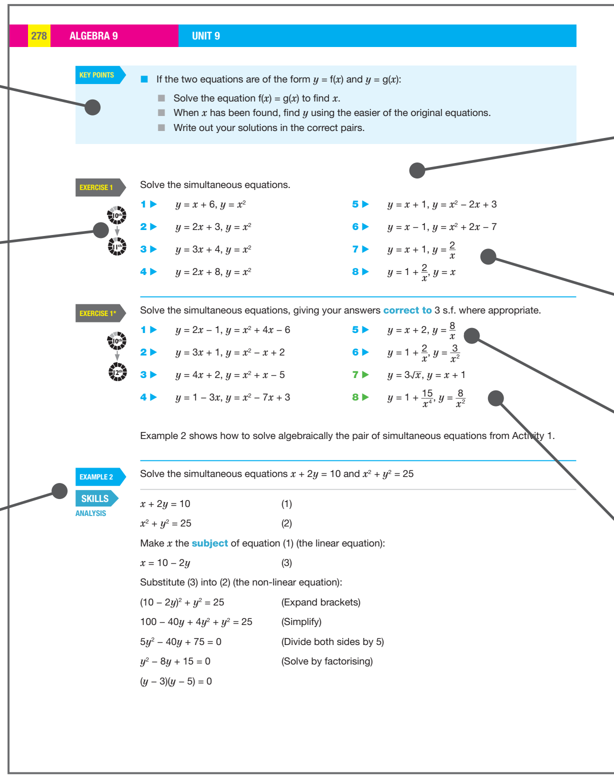
What is the connection between the line $3y = 4x - 25$ and the circle $x^2 + y^2 = 25$?

Are there any real solutions to the simultaneous equations $3y = 18 - x$ and $x^2 + y^2 = 25$?

Key Points boxes summarise the essentials.

Questions have been given a *Pearson step* from 1 to 12. This tells you how difficult the question is. The higher the step, the more challenging the question.

Examples provide a clear, instructional framework.



278 ALGEBRA 9 UNIT 9

KEY POINTS

- If the two equations are of the form $y = f(x)$ and $y = g(x)$:
  - Solve the equation $f(x) = g(x)$ to find x.
  - When x has been found, find y using the easier of the original equations.
  - Write out your solutions in the correct pairs.

EXERCISE 1

Solve the simultaneous equations.

1 ▶ $y = x + 6, y = x^2$
2 ▶ $y = 2x + 3, y = x^2$
3 ▶ $y = 3x + 4, y = x^2$
4 ▶ $y = 2x + 8, y = x^2$
5 ▶ $y = x + 1, y = x^2 - 2x + 3$
6 ▶ $y = x - 1, y = x^2 + 2x - 7$
7 ▶ $y = x + 1, y = \frac{2}{x}$
8 ▶ $y = 1 + \frac{2}{x}, y = x$

EXERCISE 1*

Solve the simultaneous equations, giving your answers **correct to** 3 s.f. where appropriate.

1 ▶ $y = 2x - 1, y = x^2 + 4x - 6$
2 ▶ $y = 3x + 1, y = x^2 - x + 2$
3 ▶ $y = 4x + 2, y = x^2 + x - 5$
4 ▶ $y = 1 - 3x, y = x^2 - 7x + 3$
5 ▶ $y = x + 2, y = \frac{8}{x}$
6 ▶ $y = 1 + \frac{2}{x}, y = \frac{3}{x^2}$
7 ▶ $y = 3\sqrt{x}, y = x + 1$
8 ▶ $y = 1 + \frac{15}{x^4}, y = \frac{8}{x^2}$

Example 2 shows how to solve algebraically the pair of simultaneous equations from Activity 1.

EXAMPLE 2 SKILLS ANALYSIS

Solve the simultaneous equations $x + 2y = 10$ and $x^2 + y^2 = 25$

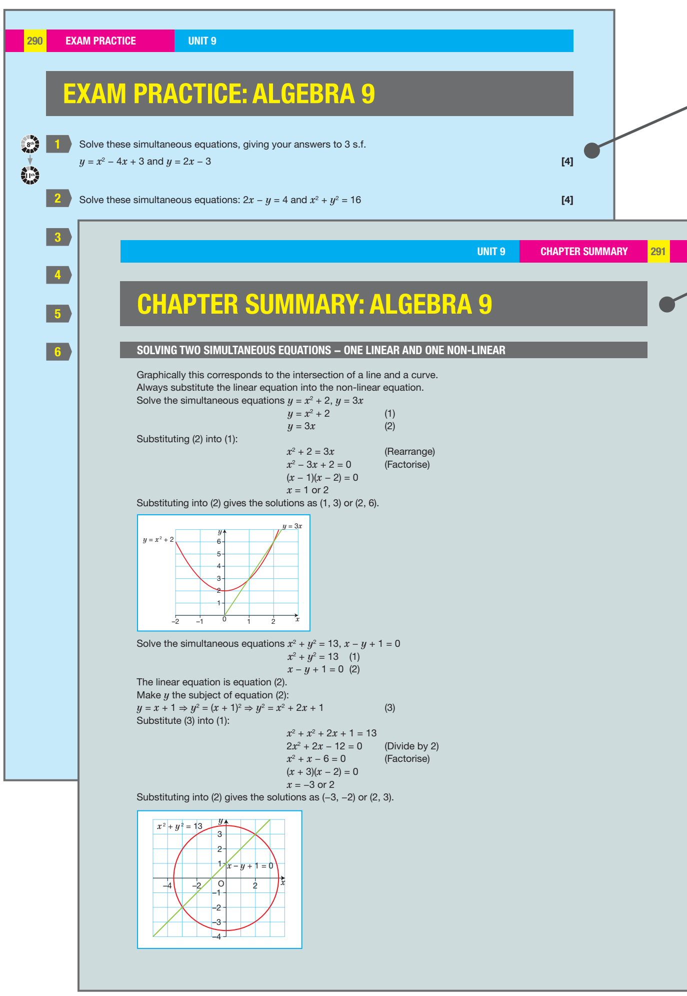
$x + 2y = 10$ (1)
$x^2 + y^2 = 25$ (2)
Make x the **subject** of equation (1) (the linear equation):
$x = 10 - 2y$ (3)
Substitute (3) into (2) (the non-linear equation):
$(10 - 2y)^2 + y^2 = 25$ (Expand brackets)
$100 - 40y + 4y^2 + y^2 = 25$ (Simplify)
$5y^2 - 40y + 75 = 0$ (Divide both sides by 5)
$y^2 - 8y + 15 = 0$ (Solve by factorising)
$(y - 3)(y - 5) = 0$

Language is graded for speakers of English as an additional language (EAL), with advanced Maths-specific terminology highlighted and defined in the glossary at the back of the book.

Non-starred exercises work towards grades 1–6 on the 9–1 scale.

Starred exercises work towards grades 6–9 on the 9–1 scale.

More difficult questions appear at the end of some exercises and are identified by green question numbers.



290 EXAM PRACTICE UNIT 9

EXAM PRACTICE: ALGEBRA 9

1 Solve these simultaneous equations, giving your answers to 3 s.f.
$y = x^2 - 4x + 3$ and $y = 2x - 3$ [4]

2 Solve these simultaneous equations: $2x - y = 4$ and $x^2 + y^2 = 16$ [4]

3

4

5

6

UNIT 9 CHAPTER SUMMARY 291

CHAPTER SUMMARY: ALGEBRA 9

SOLVING TWO SIMULTANEOUS EQUATIONS – ONE LINEAR AND ONE NON-LINEAR

Graphically this corresponds to the intersection of a line and a curve.
Always substitute the linear equation into the non-linear equation.
Solve the simultaneous equations $y = x^2 + 2, y = 3x$
$y = x^2 + 2$ (1)
$y = 3x$ (2)
Substituting (2) into (1):
$x^2 + 2 = 3x$ (Rearrange)
$x^2 - 3x + 2 = 0$ (Factorise)
$(x - 1)(x - 2) = 0$
$x = 1$ or 2
Substituting into (2) gives the solutions as (1, 3) or (2, 6).

Solve the simultaneous equations $x^2 + y^2 = 13, x - y + 1 = 0$
$x^2 + y^2 = 13$ (1)
$x - y + 1 = 0$ (2)
The linear equation is equation (2).
Make y the subject of equation (2):
$y = x + 1 \Rightarrow y^2 = (x + 1)^2 \Rightarrow y^2 = x^2 + 2x + 1$ (3)
Substitute (3) into (1):
$x^2 + x^2 + 2x + 1 = 13$
$2x^2 + 2x - 12 = 0$ (Divide by 2)
$x^2 + x - 6 = 0$ (Factorise)
$(x + 3)(x - 2) = 0$
$x = -3$ or 2
Substituting into (2) gives the solutions as (–3, –2) or (2, 3).

Exam Practice tests cover the whole chapter and provide quick, effective feedback on your progress.

Chapter Summaries state the most important points of each chapter.

EXTRA RESOURCES

Interactive practice activities and teacher support are provided online as part of Pearson's ActiveLearn Digital Service. This includes downloadable materials in the Teacher's Resource Pack for Student Books 1 and 2:

- 150 lesson plans
- 100 prior knowledge presentations and worksheets
- 90 starter activities, presentations and worksheets
- 200 videos and animations
- Pearson progression self-assessment charts.

ASSESSMENT OBJECTIVES

The following tables give an overview of the assessment for this course.

We recommend that you study this information closely to help ensure that you are fully prepared for this course and know exactly what to expect in the assessment.

PAPER 1	PERCENTAGE	MARK	TIME	AVAILABILITY
HIGHER TIER MATHS A Written examination paper Paper code 4MA1/3H Externally set and assessed by Edexcel	50%	100	2 hours	January and June examination series First assessment June 2018
PAPER 2	**PERCENTAGE**	**MARK**	**TIME**	**AVAILABILITY**
HIGHER TIER MATHS A Written examination paper Paper code 4MA1/4H Externally set and assessed by Edexcel	50%	100	2 hours	January and June examination series First assessment June 2018

ASSESSMENT OBJECTIVES AND WEIGHTINGS

ASSESSMENT OBJECTIVE	DESCRIPTION	% IN INTERNATIONAL GCSE
AO1	Demonstrate knowledge, understanding and skills in number and algebra: • numbers and the numbering system • calculations • solving numerical problems • equations, formulae and identities • sequences, functions and graphs	57–63%
AO2	Demonstrate knowledge, understanding and skills in shape, space and measures: • geometry and trigonometry • vectors and transformation geometry	22–28%
AO3	Demonstrate knowledge, understanding and skills in handling data: • statistics • probability	12–18%

ASSESSMENT SUMMARY

The Edexcel International GCSE (9–1) in Mathematics (Specification A) **Higher Tier** requires students to demonstrate application and understanding of the following topics.

NUMBER

- Use numerical skills in a purely mathematical way and in real-life situations.

ALGEBRA

- Use letters as equivalent to numbers and as variables.
- Understand the distinction between expressions, equations and formulae.
- Use algebra to set up and solve problems.
- Demonstrate manipulative skills.
- Construct and use graphs.

GEOMETRY

- Use the properties of angles.
- Understand a range of transformations.
- Work within the metric system.
- Understand ideas of space and shape.
- Use ruler, compasses and protractor appropriately.

STATISTICS

- Understand basic ideas of statistical averages.
- Use a range of statistical techniques.
- Use basic ideas of probability.

Students should also be able to demonstrate **problem-solving skills** by translating problems in mathematical or non-mathematical contexts into a process or a series of mathematical processes.

Students should be able to demonstrate **reasoning skills** by

- making deductions and drawing conclusions from mathematical information
- constructing chains of reasoning
- presenting arguments and proofs
- interpreting and communicating information accurately.

CALCULATORS

Students will be expected to have access to a suitable electronic calculator for both examination papers. The electronic calculator to be used by students attempting **Higher Tier** examination papers (3H and 4H) should have these functions as a minimum:

$+, -, \times, \div, x^2, \sqrt{x}$, memory, brackets, x^y, $x^{\frac{1}{y}}$, $\bar{x}$, Σx, Σfx, standard form, sine, cosine, tangent and their inverses.

PROHIBITIONS

Calculators with any of the following facilities are prohibited in all examinations:

- databanks
- retrieval of text or formulae
- QWERTY keyboards
- built-in symbolic algebra manipulations
- symbolic differentiation or integration.

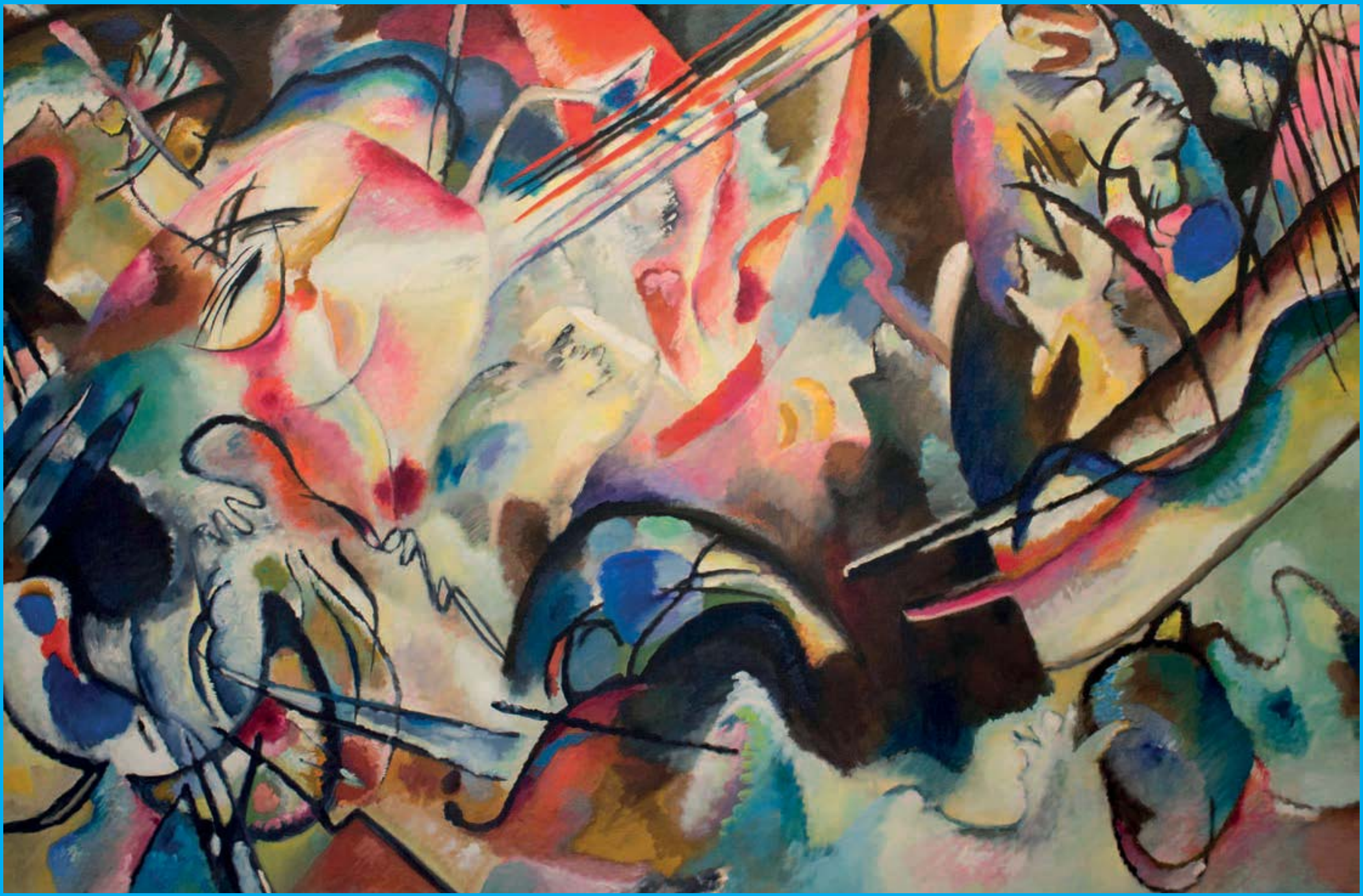
UNIT 6

The probability of throwing a six when you roll a dice is $\frac{1}{6}$

The cells of a beehive are hexagons (a polygon with 6 sides).

6 is the smallest perfect number: a number whose divisors add up to that number ($1 \times 2 \times 3 = 6$ and $1 + 2 + 3 = 6$).

The product of any three consecutive integers is always divisible by 6.



NUMBER 6

Until the early part of the 17th century, all calculations were done by long multiplication and division, which was very laborious. In 1614, Scottish mathematician John Napier showed how multiplication could be simplified using special tables called 'logarithms'. These changed the tasks of multiplying and dividing into addition and subtraction. They also simplified the calculation of powers and roots. The ideas behind these tables come from the index laws $a^m \times a^n = a^{m+n}$ and $a^m \div a^n = a^{m-n}$. Nowadays calculators can do all these tasks much faster and more accurately!

John Napier 1550–1617 ▶

LEARNING OBJECTIVES

- Recognise and use direct proportion
- Recognise and use inverse proportion
- Use index laws to simplify numerical expressions involving negative and fractional indices

BASIC PRINCIPLES

- Plot curved graphs.
- Make a table of values, plotting the points and joining them using a smooth curve.
- Rules of **indices**:
  - When multiplying, add the indices: $x^m \times x^n = x^{m+n}$
  - When dividing, subtract the indices: $x^m \div x^n = x^{m-n}$
  - When raising to a **power**, multiply the indices: $(x^m)^n = x^{mn}$
- Express numbers in **prime factor** form: $72 = 8 \times 9 = 2^3 \times 3^2$
- Convert metric units of length.
- Understand **ratio**.

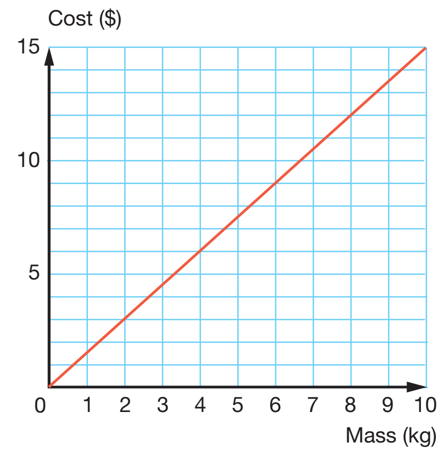
DIRECT PROPORTION

If two quantities are in **direct proportion**, then when one is multiplied or divided by a number, so is the other.

For example, if 4 kg of apples cost \$6, then 8 kg cost \$12, 2 kg cost \$3 and so on.

This relationship produces a straight-line graph through the origin.

When two quantities are in direct proportion, the graph of the relationship will always be a straight line through the origin.



EXAMPLE 1

SKILLS

PROBLEM SOLVING

The cost of a phone call is directly proportional to its length. A three-minute call costs \$4.20.

a What is the cost of an eight-minute call?

b A call costs \$23.10. How long is it?

a 3 minutes costs \$4.20

$\Rightarrow$ 1 minute costs \$4.20 ÷ 3 = \$1.40

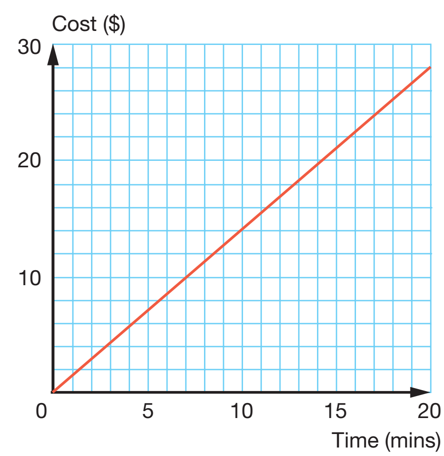
$\Rightarrow$ 8 minutes costs 8 × \$1.40 = \$11.20

b 1 minute costs \$1.40

$\Rightarrow$ \$1 gives $\frac{1}{1.40}$ minutes

$\Rightarrow$ \$23.10 gives $23.10 \times \frac{1}{1.40}$ minutes = 16.5 minutes

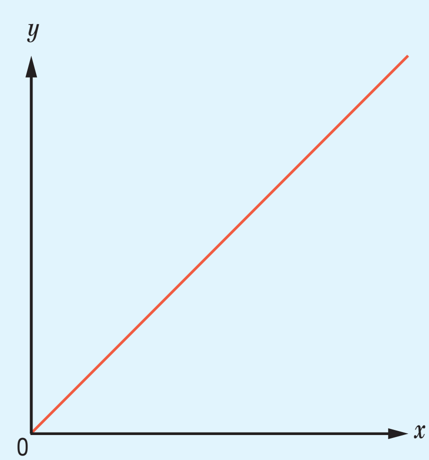
In Example 1, the graph of cost plotted against the number of minutes is a straight line through the origin.



KEY POINTS

If two quantities are in direct proportion:

- when one is multiplied or divided by a number, so is the other
- their ratio stays the same as they increase or decrease
- the graph of the relationship will always be a straight line through the origin.



EXERCISE 1

5th → 7th

1 ▶ Are these pairs of quantities in direct proportion?
Give reasons for your answers.

a 12 hotdogs cost £21.60, 15 hotdogs cost £27.00
b 5 apples cost £1.60, 9 apples cost £2.97
c Tom took 45 minutes to run 10 km, Ric took 1 hour 2 minutes to run 14 km

2 ▶ The table shows the distance travelled by a car over a period of time.

DISTANCE, s (km)	3	6	15	24
TIME, t (minutes)	4	8	20	32

a Is s in direct proportion to t? Give a reason for your answer.
b What is the formula connecting s and t?
c Work out the distance travelled after 24 minutes.

3 ▶ A band of four people can play a song in 4 minutes 30 seconds. How long does it take a band of eight people to play the same song?

4 ▶ In a science experiment, the speed of a ball bearing is measured at different times as it rolls down a slope. The table shows the results.

TIME, t (s)	1	2	3	4	5
SPEED, v (m/s)	0.98	1.96	2.94	3.92	4.9

Are time and speed in direct proportion? Give reasons for your answer.

5 ▶ The time it takes a kettle to boil some water, t, is directly proportional to the volume, V, of water in the kettle.

a Copy and complete the table.

t (seconds)	60		140		260
V (cm^3)		500	700	1000	

b What is the formula connecting V and t?
c How many minutes will it take to boil 1.5 litres of water?

6 ▶ A queen termite lays 1 million eggs in 18 days.

a How many eggs does the queen lay in one day?
b How many eggs does the queen lay in one minute?

State any assumptions you make.

7 ▶ 4 tickets to a theme park cost $104.

a How much will 18 tickets cost?
b Jo pays $182 for some tickets.
How many did she buy?

8 ▶ The cost of wood is directly proportional to its length. A 3000 mm length costs $58.50.

a Find the cost of a 5 m length of this wood.
b Find the length of a piece of this wood that costs $44.85.

EXERCISE 1*

5th → 7th

1 ▶ Seven girls take 3 min 30 s to do a dance routine for an exam.
How long will it take a group of nine girls to do the same routine?

2 ▶ The table shows some lengths in both miles and kilometres.

MILES	2.5	8	12.5	14	17.5
KILOMETRES	4	12.8	20	22.4	28

a Are miles and kilometres in direct proportion?
Give a reason for your answer.
b What is the ratio of miles : kilometres in the form $1 : n$?
c How many miles is 100 kilometres?

3 ▶ Kishan measures the extension of a spring when different weights are added to it. The table shows the results.

WEIGHT, w (N)	2.5	3.5	4.5	5.5	6.5
EXTENSION, e (mm)	11.25	15.75	22.05	24.75	29.25

The weight and extension are in direct proportion, but Kishan made one mistake in recording his results. Find and correct the mistake.

4 ▶ The local supermarket is offering an exchange rate of 1.2585 euros for £1. Chas receives 377.55 euros in a transaction. How many £ did Chas exchange?

5 ▶ The cost of a newspaper advertisement is directly proportional to the area of the advertisement.

a Copy and complete the table.

AREA, A (cm^2)		30		70	100
COST, C ($)	1125		1800	3150	

b Find the formula connecting C and A.
c An advertisement costs $3825. What is the area of the advertisement?

6 ▶ The length of the shadow of an object is directly proportional to its height. A 4.8 m tall lamp post has a shadow 2.1 m long.

a Find the height of a nearby bus stop with a shadow 1.05 m long.
b A nearby church spire is 30.4 m tall. Find the length of its shadow.

7 ▶ A recipe for chocolate cheesecake that serves 6 people uses 270 g of chocolate.

a Naomi wants to make a cheesecake for 8 people. How much chocolate should she buy?
b The chocolate is only available in 450 g bars. If she uses all the chocolate in the cheesecake, and scales the other ingredients accordingly, how many people will the cheesecake serve?

8 ▶ The resistance, R ohms, of some wire is directly proportional to its length, l mm. A piece of wire 600 mm long has a resistance of 2.1 ohms.

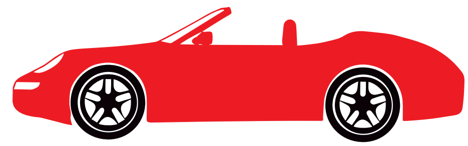
a Find the resistance of a 1.5 m length of this wire.
b Find the formula connecting R and l.
c Tamas needs to make a 7.7 ohm resistor. How many metres of wire does he need?

INVERSE PROPORTION

ACTIVITY 1

SKILLS
MODELLING

Ethan and Mia are planning a journey of 120 km. They first work out the time it will take them travelling at various speeds.



Copy and complete the table showing their results.

TIME, t (hours)	2	3	4	5		8
SPEED, v (km/hr)		40			20	
$t \times v$		120				

Plot their results on a graph of v km/hr against t hours for $0 \leq t \leq 8$

Use your graph to find the speed required to do the journey in $2\frac{1}{2}$ hours.

Is there an easier way to find this speed?

The quantities (time and speed) in Activity 1 are in **inverse proportion**.

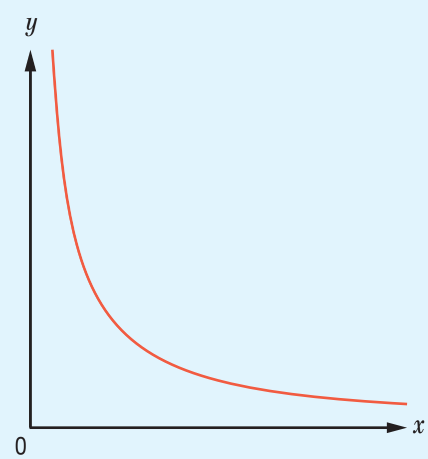
The graph you drew in Activity 1 is called a **reciprocal graph**.

In Activity 1 when the two quantities are multiplied together the answer is always 120. The **product** of two quantities is always constant if the quantities are in inverse proportion. This makes it easy to calculate values.

KEY POINTS

When two quantities are in inverse proportion:

- the graph of the relationship is a reciprocal graph
- one quantity increases at the same **rate** as the other quantity decreases, for example, as one doubles (× 2) the other halves (÷ 2)
- their product is constant.



EXAMPLE 2

SKILLS
MODELLING

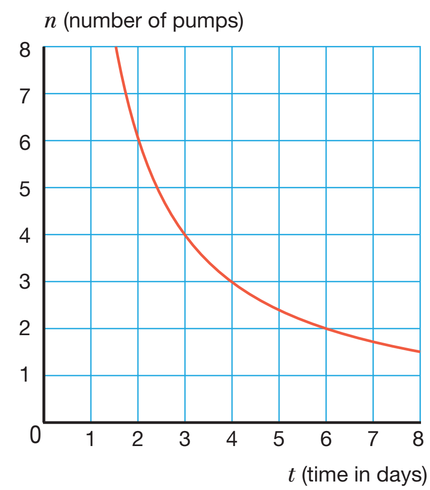
The emergency services are planning to pump out a flooded area. The number of pumps needed, n, is in inverse proportion to the time taken in days, t. It will take 6 days using two pumps.

a How long will it take using 4 pumps?
b How many pumps are needed to do the job in 2 days?

The quantities are in inverse proportion, so $n \times t$ is constant.
When $n = 2$, $t = 6 \Rightarrow nt = 12$

a If $n = 4$ then $t = 3$ (as $4 \times 3 = 12$) so it will take 3 days.
b If $t = 2$ then $n = 6$ (as $6 \times 2 = 12$) so six pumps are needed.

If the number of pumps is plotted against time a reciprocal graph will be produced.



EXERCISE 2

7th → 10th

1 ▶ The time it takes an object to travel a fixed distance is shown in the table.

TIME, t (seconds)	4	8	10	15
SPEED, s (m/s)	150	75	60	40

a Are time and speed inversely proportional? Give a reason for your answer.
b If the time taken is 12 seconds, what is the speed?

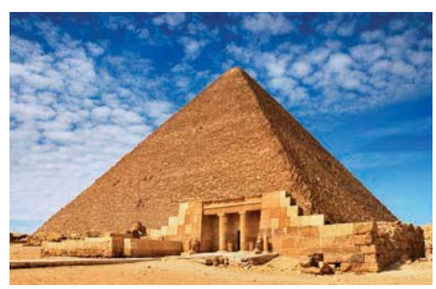
2 ▶ The number of burgers sold each day by Ben's Burger Bar is inversely proportional to the temperature.

a Copy and complete this table.

TEMPERATURE, T (°C)	10			25	30
NUMBER SOLD, N		375	300	240	

b Find the formula connecting T and N.
c Plot a graph to show this data.
d Comment on predicted sales as the temperature approaches 0 °C.

3 ▶ It has been estimated that it took 4000 men 30 years to build the largest pyramid at Giza, in Egypt, over 4500 years ago.



Assuming the number of men and the time taken are in inverse proportion,
a How long would it have taken with 6000 men?
b If a **similar** pyramid needed to be built in 10 years, how many men would be required?

4 ▶ Ava finds the time, t seconds, taken to download a music video is inversely proportional to her internet connection speed, s Mb/s. It takes 2 seconds when the speed is 10 Mb/s.

a How long will it take if the internet connection speed is 16 Mb/s?
b What must the internet connection speed be to download it in 0.5 s?
c Find the formula connecting s and t.

5 ▶ The Forth Bridge is a famous bridge over the Firth of Forth in Scotland. It was recently repainted, and it took a team of 400 painters 10 years to complete.

a If 250 painters had been used, how long would it have taken?

b How many painters would have been needed to complete the work in 8 years?

6 ▶ The food in Luke's bird feeder lasts for 3 days when 200 birds visit it per day. Luke thinks these two quantities are inversely proportional.

a Assuming Luke is correct, how many days will the food last if only 150 birds visit it per day?

b When 300 birds visit per day, the food lasts for 2 days. Is Luke's assumption correct? Give a reason for your answer.

EXERCISE 2*

7th → 10th

1 ▶ Two quantities, A and B, are in inverse proportion.

a Copy and complete the table.

A	3		5		32
B		10	9.6	8	

b Find the formula connecting A and B.

2 ▶ There are 3 trillion trees on Earth and the concentration of carbon dioxide in the atmosphere is 400 parts per million. Assuming the number of trees is inversely proportional to the concentration of carbon dioxide, find

a the concentration of carbon dioxide if the number of trees falls to 2 trillion

b the number of extra trees needed to reduce the concentration of carbon dioxide to 300 parts per million.

3 ▶ Bella's food mixer has 10 speed settings. When the setting is at 6, it takes her 4 minutes to whip some cream. Bella thinks the speed setting is inversely proportional to the time taken.

a If Bella is correct, how long will it take her to whip the cream at a speed setting of 8?

b When Bella whips cream at a speed setting of 4, it takes 5 minutes. Is the speed setting inversely proportional to the time taken? Give a reason for your answer.

4 ▶ In an electrical circuit, the current (A amps) is inversely proportional to the resistance (R ohms). The current is 9 amps when the resistance is 16 ohms.

a Find the current when the resistance is 12 ohms.

b Find the formula connecting A and R.

c The current must be limited to 6 amps. What resistance will achieve this?

5 ▶ The Shard in London is 306 m high, making it the tallest building in Europe. It has 11 000 panes of glass which take a team of 6 window cleaners 20 days to clean. The window cleaners have to abseil down the outside of the building to do this.

a If one of the cleaners falls ill, how long will it take to clean the windows?
b How many cleaners would be needed to clean the windows in 15 days?

6 ▶ Sienna is training for a charity cycle ride. When she averages 24 km/hr she takes 2 hrs 15 mins to do the course.

a When she started training she took 2 hrs 42 mins. What was her average speed?
b When Sienna finally did the ride she averaged 27.5 km/hr. How long, to the nearest minute, did it take her?

ACTIVITY 2

SKILLS
MODELLING

The speed of a professional tennis player's serve is around 250 km/hr. Could you return it?

The speed of the serve is inversely related to the reaction time needed. When the speed of serve is 250 km/hr, the reaction time needed is 0.25 s.

To work out your reaction time, do the following experiment with a 30 cm ruler.

- Ask a friend to hold a ruler near the 30 cm mark, with the ruler hanging vertically.
- Without touching the ruler, place your index finger and thumb either side of the 0 cm mark.
- Your friend must now let go of the ruler without warning, and you must catch it as soon as possible.
- Record the reading (x cm) where you catch the ruler.

1 ▶ Use the formula $t = \sqrt{\frac{x}{490}}$ seconds, to convert x into your reaction time.

2 ▶ Calculate a better estimate of your reaction time by repeating the experiment five times to find the **mean** time.

3 ▶ What speed of serve could you return?

FRACTIONAL INDICES

The laws of indices can be extended to fractional indices.

ACTIVITY 3

SKILLS
REASONING

a Use your calculator to work out:

i $4^{\frac{1}{2}}$ **ii** $9^{\frac{1}{2}}$ **iii** $16^{\frac{1}{2}}$ **iv** $25^{\frac{1}{2}}$ **v** $36^{\frac{1}{2}}$

What does this suggest $a^{\frac{1}{2}}$ means?

Check your theory with some other numbers.

b Use your calculator to work out:

i $8^{\frac{1}{3}}$ **ii** $27^{\frac{1}{3}}$ **iii** $64^{\frac{1}{3}}$

What does this suggest $a^{\frac{1}{3}}$ means?

Check your theory with some other numbers.

Using the laws of indices, $4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 4^{\left(\frac{1}{2}+\frac{1}{2}\right)} = 4^1 = 4$

But $\sqrt{4} \times \sqrt{4} = 4$

This means that $4^{\frac{1}{2}} = \sqrt{4} = 2$ so $4^{\frac{1}{2}}$ means the square root of 4

Similarly, $8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} = 8^{\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)} = 8^1 = 8$

But $\sqrt[3]{8} \times \sqrt[3]{8} \times \sqrt[3]{8} = 8$

This means that $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$ so $8^{\frac{1}{3}}$ means the cube root of 8

In a similar way it can be shown that $5^{\frac{1}{4}} = \sqrt[4]{5}$, $7^{\frac{1}{5}} = \sqrt[5]{7}$ and so on.

Writing numbers as powers can simplify the **working**.

EXAMPLE 3

Work out $8^{\frac{1}{3}}$

Method 1: Using $x^{\frac{1}{m}} = \sqrt[m]{x}$

$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

Method 2: Write 8 as a power of 2, then use the index laws to simplify.

$8^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2^{3\times\frac{1}{3}} = 2^1 = 2$

EXAMPLE 4

Work out $8^{\frac{2}{3}}$

Method 1: Using $x^{\frac{1}{m}} = \sqrt[m]{x}$

$8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

Method 2: Write 8 as a power of 2, then use the index laws to simplify.

$8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^{3\times\frac{2}{3}} = 2^2 = 4$

As $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2$, $8^{\frac{2}{3}}$ can be thought of as the square of the cube root of 8.

Alternatively, as $8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}}$ it can be thought of as the cube root of 8 squared.

EXAMPLE 5

Work out $\sqrt[3]{27^2}$

Method 1: Using $x^{\frac{n}{m}} = (\sqrt[m]{x})^n$

$\sqrt[3]{27^2} = (27^2)^{\frac{1}{3}} = 27^{\frac{2}{3}} = (\sqrt[3]{27})^2 = 3^2 = 9$

Method 2: Write 27 as a power of 3 and write the cube root as a fractional **index**.

$\sqrt[3]{27^2} = [(3^3)^2]^{\frac{1}{3}} = 3^{3\times2\times\frac{1}{3}} = 3^2 = 9$

EXAMPLE 6

Work out $\left(\frac{4}{25}\right)^{\frac{3}{2}}$

Method 1: Using $x^{\frac{n}{m}} = (\sqrt[m]{x})^n$

$\left(\frac{4}{25}\right)^{\frac{3}{2}} = \left(\sqrt{\frac{4}{25}}\right)^3 = \left(\frac{2}{5}\right)^3 = \frac{8}{125}$

Method 2: Write the numbers as powers, then use the index laws to **simplify**.

$\left(\frac{4}{25}\right)^{\frac{3}{2}} = \left(\frac{2^2}{5^2}\right)^{\frac{3}{2}} = \frac{2^{2\times\frac{3}{2}}}{5^{2\times\frac{3}{2}}} = \frac{2^3}{5^3} = \frac{8}{125}$

SKILLS
REASONING

ACTIVITY 4

Use your calculator to check Examples 3, 4, 5 and 6.

KEY POINTS

- Use the rules:
  - $x^{\frac{1}{m}} = \sqrt[m]{x}$
  - $x^{\frac{n}{m}} = (\sqrt[m]{x})^n = \sqrt[m]{x^n}$
- Or, write the numbers in prime factor form first.

EXERCISE 3

Work out

1 ▶ $25^{\frac{1}{2}}$

2 ▶ $27^{\frac{1}{3}}$

3 ▶ $16^{\frac{1}{4}}$

4 ▶ $\left(\frac{1}{4}\right)^{\frac{1}{2}}$

5 ▶ $9^{\frac{3}{2}}$

6 ▶ $8^{\frac{5}{3}}$

7 ▶ $\left(\frac{16}{25}\right)^{\frac{1}{2}}$

8 ▶ $\left(\frac{4}{9}\right)^{\frac{3}{2}}$

9 ▶ $\left(\frac{27}{8}\right)^{\frac{2}{3}}$

10 ▶ $4^{\frac{5}{2}} \times 64^{\frac{1}{3}}$

11 ▶ $3^{\frac{1}{3}} \times 9^{\frac{1}{3}}$

12 ▶ $4^{\frac{2}{3}} \div 2^{\frac{1}{3}}$

Solve for x

13 ▶ $\sqrt[3]{20} = 20^x$

14 ▶ $3^x = (\sqrt{3})^5$

EXERCISE 3*

Work out

1 ▶ $144^{\frac{1}{2}}$

2 ▶ $900^{\frac{1}{2}}$

3 ▶ $\left(\frac{1}{8}\right)^{\frac{1}{3}}$

4 ▶ $\left(\frac{1}{32}\right)^{\frac{1}{5}}$

5 ▶ $81^{\frac{3}{4}}$

6 ▶ $(-125)^{\frac{2}{3}}$

7 ▶ $\left(\frac{64}{49}\right)^{\frac{1}{2}}$

8 ▶ $\left(\frac{81}{256}\right)^{\frac{3}{4}}$

9 ▶ $\left(-3\frac{3}{8}\right)^{\frac{4}{3}}$

10 ▶ $16^{\frac{2}{5}} \times 4^{\frac{1}{5}}$

11 ▶ $25^{\frac{3}{5}} \div 5^{\frac{1}{5}}$

12 ▶ $8^{\frac{5}{3}} \div \left(\frac{16}{25}\right)^{\frac{3}{2}}$

Solve for x

13 ▶ $7^x = \left(\sqrt[4]{7}\right)^3$

14 ▶ $\sqrt[5]{5} = 25^x$

NEGATIVE INDICES

The rules of indices can also be extended to work with negative indices.

In an earlier number chapter you used $10^{-2} = \frac{1}{10^2}$, $10^{-3} = \frac{1}{10^3}$ and so on.

This also works with numbers other than 10.

EXAMPLE 7

Show that $2^{-2} = \frac{1}{2^2}$

$2^2 \div 2^4 = 2^{-2}$ (Using $x^m \div x^n = x^{m-n}$)

But $2^2 \div 2^4 = \frac{2^2}{2^4} = \frac{1}{2^2} \Rightarrow 2^{-2} = \frac{1}{2^2}$

In a similar way it can be shown that $2^{-3} = \frac{1}{2^3}$, $5^{-4} = \frac{1}{5^4}$ and so on.

Note that $2^{-1} = \frac{1}{2^1} = \frac{1}{2}$

EXAMPLE 8

Work out $8^{-\frac{2}{3}}$

$8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{4}$ (Using the result of Example 4)

EXAMPLE 9

Work out $\left(\frac{2}{3}\right)^{-2}$

$\left(\frac{2}{3}\right)^{-2} = \frac{1}{\left(\frac{2}{3}\right)^2} = 1 \times \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

Example 9 shows that $\left(\frac{x}{y}\right)^{-n} = \frac{1}{\left(\frac{x}{y}\right)^n} = \left(\frac{y}{x}\right)^n$

EXAMPLE 10

Work out $\left(\frac{16}{81}\right)^{-\frac{3}{4}}$

$\left(\frac{16}{81}\right)^{-\frac{3}{4}} = \left(\frac{81}{16}\right)^{\frac{3}{4}} = \frac{\left(\sqrt[4]{81}\right)^3}{\left(\sqrt[4]{16}\right)^3} = \frac{3^3}{2^3} = \frac{27}{8}$

EXAMPLE 11

Show that $2^0 = 1$

$2^3 \div 2^3 = 2^0$ (Using $x^m \div x^n = x^{m-n}$)

But $2^3 \div 2^3 = \frac{2^3}{2^3} = 1 \Rightarrow 2^0 = 1$

In a similar way it can be shown that $3^0 = 1$, $4^0 = 1$, and so on.

KEY POINTS

- $x^{-n} = \frac{1}{x^n}$ for any number n, $x \neq 0$
- $\left(\frac{x}{y}\right)^{-n} = \frac{1}{\left(\frac{x}{y}\right)^n} = \left(\frac{y}{x}\right)^n$, $x, y \neq 0$
- $x^{-1} = \frac{1}{x}$, $x \neq 0$
- $x^0 = 1$, $x \neq 0$

EXERCISE 4

Work out

1 ▶ 3^{-2}

2 ▶ 2^{-3}

3 ▶ 4^{-1}

4 ▶ $(3^{-2})^{-1}$

5 ▶ $3^{-2} \div 3^{-1}$

6 ▶ 5^0

7 ▶ $\left(\frac{1}{2}\right)^{-1}$

8 ▶ $\left(\frac{3}{4}\right)^{-1}$

9 ▶ $\left(\frac{1}{3}\right)^{-4}$

10 ▶ $\left(\frac{2}{3}\right)^{-3}$

11 ▶ $\left(\frac{5}{3}\right)^{-2}$

12 ▶ $8^{-\frac{1}{3}}$

13 ▶ $25^{-\frac{1}{2}}$

14 ▶ $16^{-\frac{3}{4}}$

15 ▶ $1^{-\frac{3}{2}}$

16 ▶ $\left(\frac{1}{4}\right)^{-\frac{1}{2}}$

17 ▶ $\left(\frac{1}{8}\right)^{-\frac{2}{3}}$

18 ▶ $\left(\frac{4}{9}\right)^{-\frac{1}{2}}$

19 ▶ $\frac{4^{-3} \times 4^{-1}}{4^{-4}}$

20 ▶ $27^{-\frac{1}{3}} \times 9^{\frac{3}{2}}$

Solve for x

21 ▶ $\frac{1}{7} = 7^x$

22 ▶ $\left(\frac{1}{4}\right)^x = 4$

23 ▶ $\left(\frac{3}{2}\right)^x = \frac{2}{3}$

24 ▶ $\left(\frac{2}{3}\right)^x = \frac{9}{4}$

25 ▶ $\left(\frac{16}{25}\right)^x = \frac{5}{4}$

EXERCISE 4*

Work out

1 ▶ 5^{-2}

2 ▶ 4^{-3}

3 ▶ 12^{-1}

4 ▶ $\left(\frac{1}{4}\right)^{-1}$

5 ▶ $\left(\frac{2}{7}\right)^{-2}$

6 ▶ $\left(\frac{4}{5}\right)^{-3}$

7 ▶ $\left(1\frac{2}{3}\right)^{-2}$

8 ▶ $64^{-\frac{1}{2}}$

9 ▶ $16^{-\frac{1}{4}}$

10 ▶ $8^{-\frac{5}{3}}$

11 ▶ $1^{-\frac{4}{3}}$

12 ▶ $\left(\frac{1}{81}\right)^{-\frac{1}{2}}$

13 ▶ $\left(\frac{1}{125}\right)^{-\frac{2}{3}}$

14 ▶ $\left(\frac{27}{8}\right)^{-\frac{4}{3}}$

15 ▶ $\left(\frac{162}{32}\right)^{-\frac{3}{4}}$

16 ▶ $2 \div 32^{-\frac{1}{5}}$

17 ▶ $\left[\left(\frac{5}{7}\right)^{-\frac{2}{3}}\right]^{-3}$ **18 ▶** $\frac{7^{-2} \times 7^{-3}}{7^{-5}}$ **19 ▶** $\left(\frac{4}{25}\right)^{-\frac{1}{2}} \times \left(\frac{8}{27}\right)^{\frac{1}{3}}$ **20 ▶** $\left(\frac{27}{125}\right)^{-\frac{2}{3}} \div \left(\frac{81}{64}\right)^{-\frac{1}{2}}$

Solve for x

21 ▶ $\left(\frac{4}{5}\right)^{x} = \frac{25}{16}$ **23 ▶** $(\sqrt[3]{5})^{4} = 25^{x}$ **25 ▶** $8^{-\frac{8}{3}} = \left(\frac{1}{4}\right)^{x}$

22 ▶ $\left(\frac{49}{64}\right)^{x} = \frac{8}{7}$ **24 ▶** $\left(\frac{1}{8}\right)^{12} = 16^{x}$

EXERCISE 5

6th → 10th

REVISION

1 ▶ The table gives the relationship between two **variables**.

VARIABLE A	3	5	8	9	13
VARIABLE B	12	20	32	36	51

Are A and B in direct proportion? Give a reason for your answer.

2 ▶ The pressure of water on an object is directly proportional to its depth.

a Copy and complete the table.

DEPTH, d (metres)	5		12		40
PRESSURE, P (bars)		0.8	1.2	2.5	

b Find the formula connecting pressure and depth.

c A diver's watch has been guaranteed to work at a pressure up to 8.5 bars. A diver takes the watch down to 75 m. Will the watch still work? Give a reason for your answer.

3 ▶ The weight of a steel pipe is directly proportional to its length. A 3 m length of pipe weighs 45.75 kg.

a Find the weight of a 10 m length of this pipe.

b A piece of this pipe weighs 122 kg. How long is it?

4 ▶ The number of workers needed to install the seats in a stadium is inversely proportional to the number of days it takes.

a Copy and complete the table.

NUMBER OF WORKERS, w	4	8		2
NUMBER OF DAYS, d		6	8	

b Find the formula connecting d and w.

c How many workers are needed to install the seats in one day?

5 ▶ When making jewellery, Erin has to glue the stones into the setting. She finds that the time taken for the glue to set is inversely proportional to the temperature. When the temperature is 20 °C, the glue takes 24 hours to set fully.

a One day the heating in Erin's workshop fails and the temperature drops to 15 °C. How long will it take the glue to set?

b Erin needs to finish some jewellery in a hurry and needs the glue to set in 8 hours. What temperature will she need to achieve this?

6 ▶ Chuck has a large farm growing corn. When he uses all three of his combine harvesters it takes 4 days to harvest the corn. Assuming the number of combine harvesters and the days taken to harvest are in inverse proportion, find

a how long it will take to harvest if one of his combine harvesters breaks down

b how many harvesters are needed to harvest the corn in $2\frac{1}{2}$ days.

Work out

7 ▶ $36^{\frac{1}{2}}$

8 ▶ $8^{\frac{1}{3}}$

9 ▶ $81^{\frac{1}{4}}$

10 ▶ $\left(\frac{1}{9}\right)^{\frac{1}{2}}$

11 ▶ $1000^{\frac{2}{3}}$

12 ▶ $16^{\frac{3}{2}}$

13 ▶ $1^{\frac{2}{3}}$

14 ▶ $\left(\frac{64}{125}\right)^{\frac{1}{3}}$

15 ▶ 4^{-3}

16 ▶ $\left(\frac{1}{2}\right)^{-3}$

17 ▶ $\left(\frac{2}{3}\right)^{-2}$

18 ▶ $125^{-\frac{1}{3}}$

19 ▶ $27^{-\frac{2}{3}}$

20 ▶ $\left(\frac{1}{64}\right)^{-\frac{1}{2}}$

21 ▶ $\left(\frac{1}{32}\right)^{-\frac{3}{5}}$

22 ▶ $(2^2)^{-1}$

23 ▶ $2^2 \times 2^{-1}$

24 ▶ $\frac{3^{-1} \times 3^{-2}}{3^{-4}}$

25 ▶ $\frac{1}{5} \div 125^{-\frac{1}{3}}$

Solve for x

26 ▶ $\left(\sqrt[3]{2}\right)^3 = 2^x$

27 ▶ $\left(\frac{1}{3}\right)^x = 3$

28 ▶ $\left(\frac{25}{4}\right)^x = \frac{2}{5}$

EXERCISE 5*

7th → 10th

REVISION

1 ▶ The force on a mass is directly proportional to the **acceleration** of the mass.

a Copy and complete the table.

FORCE, F (N)	1.8			10.8	18
ACCELERATION, a (m/s^2)		0.7	1.2	3	

b Find the formula connecting F and a.

c A force of 90 N is applied. What is the acceleration?

2 ▶ Mira is downloading some computer games. The download time is directly proportional to the file size. A 45 MB file takes $3\frac{1}{3}$ seconds to download.

a How long will it take her to download an 81 MB file?
b One game takes 12 seconds to download. How large is the file?

3 ▶ The table gives the relationship between two variables.

VARIABLE X	40	7.2	5	4.8	2.4
VARIABLE Y	3.6	20	28.8	30	60

Are the variables in inverse proportion? Give a reason for your answer.

4 ▶ The time taken to fill a tank with water is inversely proportional to the number of pipes filling it.

a Copy and complete the table.

NUMBER OF PIPES, n		4		8	10
TIME, t (hrs)	18		3	2.25	

b Find the formula connecting n and t.
c The tank must be filled in 1.5 hrs. How many pipes are needed?

5 ▶ A world record pizza was just over 40 m in **diameter**. If it had been cut into 80 cm^2 pieces it would have fed 157 700 people.

a If it had been cut into 100 cm^2 pieces, how many people could have been fed?
b What area of piece would be needed to feed 252 320 people?

6 ▶ The size of the **exterior angle** of a regular **polygon** is inversely proportional to the number of sides. The exterior angle of a square is 90°.

a The exterior angle of a regular polygon is 9°. How many sides does it have?
b If the number of sides of a regular polygon is doubled, by what factor does the exterior angle change?

Work out

7 ▶ $121^{\frac{1}{2}}$

8 ▶ $(-125)^{\frac{1}{3}}$

9 ▶ $256^{\frac{1}{4}}$

10 ▶ $10\,000^{\frac{3}{4}}$

11 ▶ $8^{\frac{4}{3}}$

12 ▶ $\left(\frac{27}{64}\right)^{\frac{1}{3}}$

13 ▶ $1^{\frac{2}{5}}$

14 ▶ π^0

15 ▶ $\left(1\frac{61}{64}\right)^{\frac{2}{3}}$

16 ▶ 2^{-5}

17 ▶ 5^{-3}

18 ▶ $\left(\frac{1}{4}\right)^{-4}$

19 ▶ $81^{-\frac{1}{4}}$

20 ▶ $(-8)^{-\frac{1}{3}}$

21 ▶ $27^{-\frac{4}{3}}$

22 ▶ $\left(-\frac{1}{27}\right)^{-\frac{1}{3}}$

23 ▶ $\left(\frac{1}{81}\right)^{-\frac{3}{4}}$

24 ▶ $\left(\frac{81}{16}\right)^{\frac{3}{4}} \times \left(\frac{9}{25}\right)^{-\frac{3}{2}}$

25 ▶ $\left(\frac{27}{8}\right)^{-\frac{2}{3}} \div \left(\frac{81}{256}\right)^{-\frac{3}{4}}$

Solve for x

26 ▶ $\frac{4}{25} = \left(\frac{5}{2}\right)^x$

27 ▶ $\left(\frac{125}{64}\right)^x = \frac{4}{5}$

28 ▶ $\left(\frac{3}{2}\right)^{18} = \left(\frac{8}{27}\right)^x$

EXAM PRACTICE: NUMBER 6

1 The cost of fuel is directly proportional to the volume bought.

a Copy and complete the table.

VOLUME, v (litres)		15		45
COST, C (£)	8.5	17	34	

b Find the formula connecting C and v. **[4]**

2 Oliver finds that the number of pages of homework he writes is inversely proportional to the loudness of his music. When the volume control is set at 12 he writes 4 pages.

a If the volume control is set at 16, how many pages will he write?

b Oliver needs to write 6 pages to keep his teacher happy. What volume control setting should he use?

[4]

3 Work out

a $64^{\frac{1}{2}}$

b $64^{\frac{1}{3}}$

c $32^{\frac{3}{5}}$

d $\left(\frac{8}{27}\right)^{\frac{1}{3}}$

e $9^{\frac{3}{2}}$

f $16^{\frac{3}{4}}$

g $\left(2\frac{1}{4}\right)^{\frac{1}{2}}$

h 3^{-4}

i $\left(\frac{1}{3}\right)^{-1}$

j $\left(\frac{3}{2}\right)^{-3}$

k $27^{-\frac{1}{3}}$

l $8^{-\frac{2}{3}}$

m $\left(\frac{1}{36}\right)^{-\frac{1}{2}}$

n $\left(\frac{81}{16}\right)^{-\frac{3}{4}}$

o $1 \div 16^{-\frac{1}{4}}$

p $16^{-\frac{1}{2}} \div 64^{-\frac{2}{3}}$

Solve for x

q $\left(\frac{8}{27}\right)^{x} = \frac{3}{2}$ **[17]**

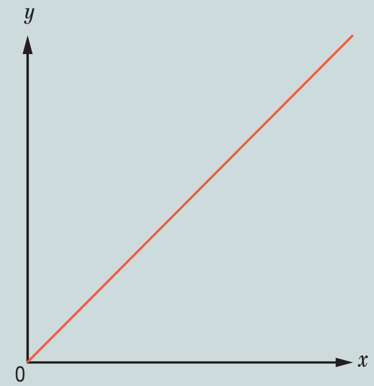
[Total 25 marks]

CHAPTER SUMMARY: NUMBER 6

DIRECT PROPORTION

If two quantities are in direct proportion

- the graph of the relationship will always be a straight line through the origin
- when one is multiplied or divided by a number, so is the other
- their ratio stays the same as they increase or decrease.



The cost of ribbon is directly proportional to its length.

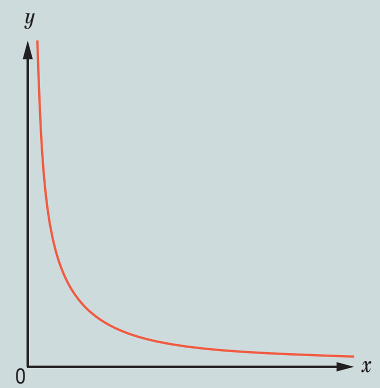
A 3.5 m piece of ribbon costs £2.38. What is the cost of 8 m of this ribbon?

1 m of ribbon costs £2.38 ÷ 3.5 = £0.68 ⇒ 8 m costs 8 × £0.68 = £5.44

INVERSE PROPORTION

If two quantities are in inverse proportion

- the graph of the relationship is a reciprocal graph
- one quantity increases at the same rate as the other quantity decreases, for example, as one doubles (× 2) the other halves (÷ 2)
- their product is constant.



A farmer has enough food for 250 chickens (c) for 24 days (d).

He buys 50 more chickens. For how many days will the food last?

The quantities are in inverse proportion
so $c \times d$ will be constant and equal to 250 × 24 = 6000

The number of chickens is now 300 ⇒ 300 × d = 6000 ⇒ d = 20 so it will last for 20 days.

FRACTIONAL INDICES

Write the numbers in prime factor form first:

- $x^{\frac{1}{m}} = \sqrt[m]{x}$ $\quad 8^{\frac{1}{3}} = \sqrt[3]{8} = 2$ $\quad$ or $\quad 8^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2^{3\times\frac{1}{3}} = 2^1 = 2$
- $x^{\frac{n}{m}} = (\sqrt[m]{x})^n = \sqrt[m]{x^n}$ $\quad 8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$ $\quad$ or $\quad 8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^{3\times\frac{2}{3}} = 2^2 = 4$

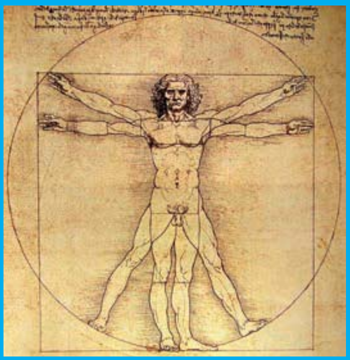
NEGATIVE INDICES

- $x^{-n} = \frac{1}{x^n}$ for any number n, $x \neq 0$ $\qquad 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
- $\left(\frac{x}{y}\right)^{-n} = \frac{1}{\left(\frac{x}{y}\right)^n} = \left(\frac{y}{x}\right)^n$, $x, y \neq 0$ $\qquad \left(\frac{3}{5}\right)^{-2} = \frac{1}{\left(\frac{3}{5}\right)^2} = \left(\frac{5}{3}\right)^2 = \frac{25}{9}$
- $x^{-1} = \frac{1}{x}$, $x \neq 0$ $\qquad 5^{-1} = \frac{1}{5}$
- $x^0 = 1$, $x \neq 0$ $\qquad 2^0 = 1$

ALGEBRA 6

The Vitruvian Man is a drawing by Leonardo da Vinci from around 1490. For example, the head measured from the forehead to the chin was exactly one tenth of the total height. It was found with notes based on the studies of the architect Vitruvius who tried to find connections between proportions found in science, nature, art and the human body.

Leonardo da Vinci 1452–1519



The Vitruvian Man

LEARNING OBJECTIVES

- Write and use formulae to solve problems involving direct proportion
- Write and use formulae to solve problems involving inverse proportion
- Use index notation involving fractional, negative and zero powers

BASIC PRINCIPLES

- Use formulae relating one **variable** to another, for example $A = \pi r^2$
- Substitute numerical values into formulae and relate the answers to applied real situations.
- Derive simple formulae (linear, quadratic and cubic).
- Know and use the laws of **indices**:

  $a^m \times a^n = a^{m+n}$

  $a^m \div a^n = a^{m-n}$

  $(a^m)^n = a^{mn}$
- Know and use the laws of indices involving fractional, negative and zero **powers** in a number context.

PROPORTION

If two quantities are related to each other, given enough information, it is possible to write a formula describing this relationship.

ACTIVITY 1

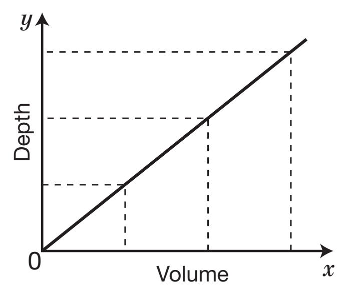
SKILLS
ANALYSIS

Copy and complete this table to show which paired items are related.

VARIABLES	RELATED? (YES OR NO)
Area of a circle A and its **radius** r	Y
Circumference of a circle C and its radius r	
Distance travelled, D, at a constant speed, x	
Mathematical ability, M, and a person's height, h	
Cost of a tin of paint, P, and its density, d	
Weight of water, W, and its volume, v	
Value of a painting, V, and its area, a	
Swimming speed, S, and collar size, x	

DIRECT PROPORTION – LINEAR

When water is poured into an empty cubical fish tank, each litre that is poured in increases the depth by a fixed amount.



A graph of depth, y, against volume, x, is a straight line through the origin, showing a linear relationship.

In this case, y is **directly proportional** to x. If y is doubled, so is x. If y is halved, so is x, and so on.

This relationship can be expressed in *any* of these ways:

- y is directly proportional to x.
- y varies directly with x.
- y varies as x.

All these statements have the same meaning.

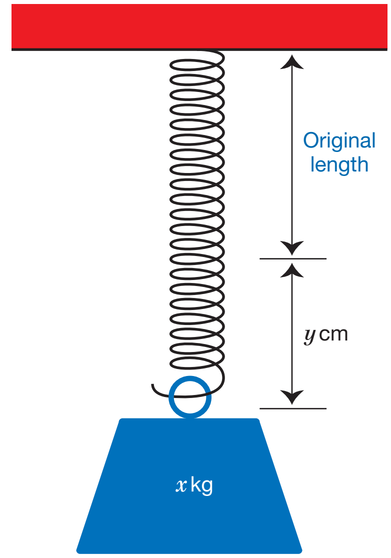
In symbols, direct proportion relationships can be written as $y \propto x$. The $\propto$ **sign** can then be replaced by '$= k$' to give the formula $y = kx$, where k is the constant of proportionality.

The graph of $y = kx$ is the equation of a straight line through the origin, with **gradient** k.

EXAMPLE 1

SKILLS
PROBLEM SOLVING

The extension, y cm, of a spring is directly proportional to the **mass**, x kg, hanging from it.



If y = 12 cm when x = 3 kg, find

a the formula for y in terms of x

b the extension y cm when a 7 kg mass is attached

c the mass x kg that produces a 20 cm extension.

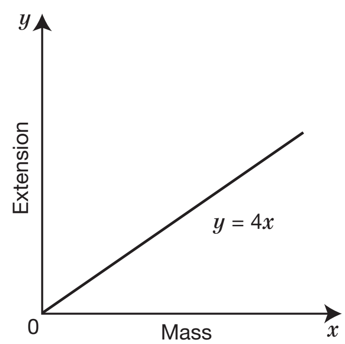
---

a y is directly proportional to x, so $y \propto x \Rightarrow y = kx$
$y = 12$ when $x = 3$, so $12 = k \times 3$
therefore, $k = 4$
Hence the formula is $y = 4x$

b When $x = 7$, $y = 4 \times 7 = 28$
The extension produced from a 7 kg mass is 28 cm.

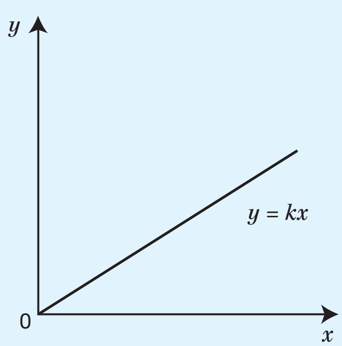
c When y = 20 cm, $20 = 4x$
therefore, $x = 5$

A 5 kg mass produces a 20 cm extension.
The graph of extension plotted against mass would look like this.



KEY POINTS

- y is directly proportional to x is written as $y \propto x$ and this means $y = kx$, for some constant k.
- The graph of y against x is a straight line through the origin.



EXERCISE 1

1 ▶ y is directly proportional to x. If $y = 10$ when $x = 2$, find

a the formula for y in terms of x

b y when $x = 6$

c x when $y = 25$

d **Sketch** the graph of y against x.

2 ▶ d is directly proportional to t. If $d = 100$ when $t = 25$, find

a the formula for d in terms of t

b d when $t = 15$

c t when $d = 180$

3 ▶ y is directly proportional to x.

$y = 15$ when $x = 3$

a Express y in terms of x.

b Find y when $x = 10$

c Find x when $y = 65$

4 ▶ y is directly proportional to x.

$y = 52$ when $x = 8$

a Write a formula for y in terms of x.

b Find y when $x = 14$

c Find x when $y = 143$

d Sketch the graph of y against x.

5 ▶ y is directly proportional to x.

a $y = 6$ when $x = 4$. Find x when $y = 7.5$

b $y = 31.5$ when $x = 7$. Find x when $y = 45.5$

c $y = 8$ when $x = 5$. Find x when $y = 13$

6 ▶ y is directly proportional to x.

When $x = 600$, $y = 10$

a Find a formula for y in terms of x.

b Calculate the value of y when $x = 540$

7 ▶ An elastic string's extension y cm varies as the mass x kg that hangs from it.

The string extends 4 cm when a 2 kg mass is attached.

a Find the formula for y in terms of x.

b Find y when $x = 5$

c Find x when $y = 15$

8 ▶ A bungee jumping rope's extension e m varies as the mass M kg of the person attached to it.

If $e = 4$ m when $M = 80$ kg, find

a the formula for e in terms of M

b the extension for a person with a mass of 100 kg

c the mass of a person when the extension is 6 m.

9 ▶ An ice-cream seller discovers that, on any particular day, the number of sales (I) is directly proportional to the temperature (t °C). 1500 sales are made when the temperature is 20 °C.

a Find the formula for I in terms of t.

b How many sales might be expected on a day with a temperature of 26 °C?

10 ▶ The number of people in a swimming pool (N) varies with the daily temperature (t °C). 175 people swim when the temperature is 25 °C.

a Find the formula for N in terms of t.

b The pool's capacity is 200 people. Will people have to queue and wait if the temperature reaches 30 °C?

EXERCISE 1*

10th

1 ▶ The speed of a stone, v m/s, falling off a cliff is directly proportional to the time, t seconds, after release. Its speed is 4.9 m/s after 0.5 s.

a Find the formula for v in terms of t.

b What is the speed after 5 s?

c At what time is the speed 24.5 m/s?

2 ▶ The cost, c cents, of a tin of salmon varies directly with its mass, m g.

The cost of a 450 g tin is 150 cents.

a Find the formula for c in terms of m.

b How much does a 750 g tin cost?

c What is the mass of a tin costing $2?

3 ▶ The distance a honey bee travels, d km, is directly proportional to the mass of the honey, m g, that it produces. A bee travels 150 000 km to produce 1 kg of honey.

a Find the formula for d in terms of m.

b What distance is travelled by a bee to produce 10 g of honey?

c What mass of honey is produced by a bee travelling once around the world – a distance of 40 000 km?

4 ▶ The mass of sugar, m g, used in making chocolate cookies varies directly with the number of cookies, n. 3.25 kg of sugar is used to make 500 cookies.

a Find the formula for m in terms of n.

b What mass of sugar is needed for 150 cookies?

c How many cookies can be made using 10 kg of sugar?

5 ▶ The height of a tree, h m, varies directly with its age, y years.

A 9 m tree is 6 years old.

a Find the formula for h in terms of y.

b What height is a tree that is 6 months old?

c What is the age of a tree that is 50 cm tall?

6 ▶ The distance, d (in km), covered by an aeroplane is directly proportional to the time taken, t (in hours).

The aeroplane covers a distance of 1600 km in 3.2 hours.

a Find the formula for d in terms of t.

b Find the value of d when $t = 5$

c Find the value of t when $d = 2250$

d What happens to the distance travelled, d, when the time, t, is

i doubled **ii** halved?

7 ▶ The cost, C (in £), of a newspaper advert is directly proportional to the area, A (in cm^2), of the advert.

An advert with an area of 40 cm^2 costs £2000.

a Sketch a graph of C against A.

b Write a formula for C in terms of A.

c Use your formula to work out the cost of an 85 cm^2 advert.

8 ▶ y is directly proportional to x.

$y = 46$ when $x = 6$

a Write a formula for y in terms of x.

b Find y when $x = 24$

c Find x when $y = 161$

9 ▶ The distance, d (in km), covered by a long-distance runner is directly proportional to the time taken, t (in hours).

The runner covers a distance of 42 km in 4 hours.

a Find a formula for d in terms of t.

b Find the value of d when $t = 8$

c Find the value of t when $d = 7.7$

d What happens to the distance travelled, d, when the time, t, is

i trebled **ii** divided by 3?

10 ▶ The amount, C (in £), that a plumber charges is directly proportional to the time, t (in hours), that the plumber works. A plumber earns £247.50 when she works 5.5 hours.

a Sketch a graph of C against t.

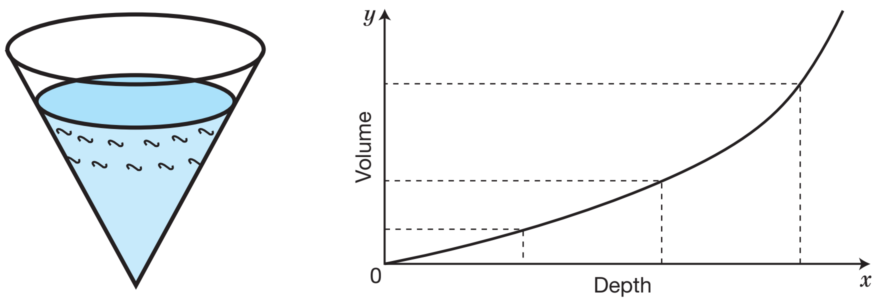
b Write a formula for C in terms of t.

c Use your formula to work out how many hours the plumber has worked when she earns £1035.

DIRECT PROPORTION – NONLINEAR

Water is poured into an empty inverted cone. Each litre poured in will result in a different depth increase.

A graph of volume, y, against depth, x, will illustrate a direct **nonlinear relationship**.



This is the graph of a **cubic** relationship.
This relationship can be expressed in *either* of these ways:

- y is directly proportional to x cubed.
- y varies as x cubed.

Both statements have the same meaning.
In symbols, this relationship is written as $y \propto x^3$
The $\propto$ sign can then be replaced by '$= k$' to give the formula $y = kx^3$, where k is the constant of proportionality.
A quantity can also be directly proportional to the square or the square root of another quantity.

EXAMPLE 2

SKILLS
ANALYSIS

Express these relationships as equations with constants of proportionality.

a y is directly proportional to x squared.

b m varies directly with the cube of n.

c s is directly proportional to the square root of t.

d v squared varies as the cube of w.

a $y \propto x^2 \Rightarrow y = kx^2$

b $m \propto n^3 \Rightarrow m = kn^3$

c $s \propto \sqrt{t} \Rightarrow s = k\sqrt{t}$

d $v^2 \propto w^3 \Rightarrow v^2 = kw^3$

KEY POINTS

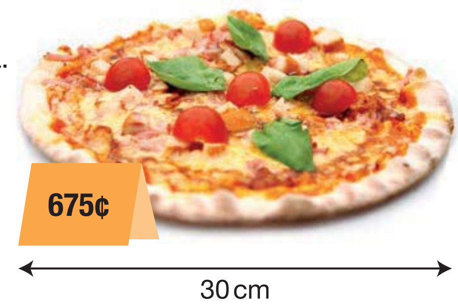
- If y is proportional to x squared then $y \propto x^2$ and $y = kx^2$, for some constant k.
- If y is proportional to x cubed then $y \propto x^3$ and $y = kx^3$, for some constant k.
- If y is proportional to the square root of x then $y \propto \sqrt{x}$ and $y = k\sqrt{x}$, for some constant k.

EXAMPLE 3

SKILLS

PROBLEM SOLVING

The cost of Luciano's take-away pizzas (C cents) is directly proportional to the square of the **diameter** (d cm) of the pizza. A 30 cm pizza costs 675 cents.



a Find a formula for C in terms of d and use it to find the price of a 20 cm pizza.

b What size of pizza can you expect for \$4.50?

---

a C is proportional to d^2, so $C \propto d^2$ — $C = kd^2$

$C = 675$ when $d = 30$ — $675 = k(30)^2$

$k = 0.75$

The formula is therefore $C = 0.75d^2$

When $d = 20$ — $C = 0.75(20)^2$

$C = 300$

The cost of a 20 cm pizza is 300 cents (\$3).

b When $C = 450$ — $450 = 0.75d^2$

$d^2 = 600$

$d = \sqrt{600} = 24.5$ (3 s.f.)

A \$4.50 pizza should be 24.5 cm in diameter.

EXERCISE 2

9th → 10th

1 ▶ y is directly proportional to the square of x. If $y = 100$ when $x = 5$, find

a the formula for y in terms of x

b y when $x = 6$

c x when $y = 64$

2 ▶ p varies directly as the square of q. If $p = 72$ when $q = 6$, find

a the formula for p in terms of q

b p when $q = 3$

c q when $p = 98$

3 ▶ v is directly proportional to the cube of w. If $v = 16$ when $w = 2$, find

a the formula for v in terms of w

b v when $w = 3$

c w when $v = 128$

4 ▶ m varies directly as the square root of n. If $m = 10$ when $n = 1$, find

a the formula for m in terms of n

b m when $n = 4$

c n when $m = 50$

5 ▶ The distance fallen by a parachutist, y m, is directly proportional to the square of the time taken, t secs. If 20 m are fallen in 2 s, find

a the formula expressing y in terms of t

b the distance fallen through in 3 s

c the time taken to fall 100 m.

6 ▶ 'Espirit' perfume is available in bottles of different volumes of **similar** shapes. The price, $\$P$, is directly proportional to the cube of the bottle height, h cm. A 10 cm high bottle is \$50. Find

a the formula for P in terms of h

b the price of a 12 cm high bottle

c the height of a bottle of 'Espirit' costing \$25.60

7 ▶ The kinetic energy, E (in joules, J), of an object varies in direct proportion to the square of its speed, s (in m/s). An object moving at 5 m/s has 125 J of kinetic energy.

a Write a formula for E in terms of s.

b How much kinetic energy does the object have if it is moving at 2 m/s?

c What speed is the object moving at if it has 192.2 J of kinetic energy?

d What happens to the kinetic energy, E, if the speed of the object is doubled?

8 ▶ The cost of fuel per hour, C (in £), to move a boat through the water is directly proportional to the cube of its speed, s (in mph).

A boat travelling at 10 mph uses £50 of fuel per hour.

a Write a formula for C in terms of s.

b Calculate C when the boat is travelling at 5 mph.

EXERCISE 2*

10th

1 ▶ f is directly proportional to g^2. Copy and complete this table.

g	2	4	
f	12		108

2 ▶ m is directly proportional to n^3. Copy and complete this table.

n	1		5
m	4	32	

3 ▶ In a factory, chemical reactions are carried out in spherical containers. The time, T (in minutes), the chemical reaction takes is directly proportional to the square of the radius, R (in cm), of the spherical container.

When $R = 120$, $T = 32$

a Write a formula for T in terms of R.

b Find the value of T when $R = 150$

4 ▶ When an object accelerates steadily from rest, the distance, d (in metres), that it travels varies in direct proportion to the square of the time, t (in seconds), that it has been travelling.

An object moves 176.4 m in 6 seconds.

a Write a formula for d in terms of t.

b How far does an object move if it accelerates like this for 10 seconds from rest?

c How many seconds has an object been accelerating for, if it has moved 1102.5 m?

d What happens to the distance moved, d, if the time for which an object has been accelerating is doubled?

5 ▶ The volume, V (in cm^3), of a sphere is directly proportional to the cube of its radius, r (in cm). A sphere with a radius of 5 cm has a volume of 523.5 cm^3.

a Write a formula for V in terms of r.
b Calculate V when the radius is 20 cm.
c Sketch the graph of V against r.

6 ▶ The resistance to motion, R newtons, of the 'Storm' racing car is directly proportional to the square of its speed, s km/hour. When the car travels at 160 km/hour it experiences a 500 newton resistance.

a Find the formula for R in terms of s.
b What is the car's speed when it experiences a resistance of 250 newtons?

7 ▶ The height of giants, H metres, is directly proportional to the cube root of their age, y years. An 8-year-old giant is 3 m tall.

a Find the formula for H in terms of y.
b What age is a 12 m tall giant?

8 ▶ The surface area of a sphere is directly proportional to the square of its radius.
A sphere of radius 10 cm must be increased to a radius of x cm if its surface area is to be doubled. Find x.

ACTIVITY 2

SKILLS
ANALYSIS

The German astronomer Kepler (1571–1630) created three astronomical laws. Kepler's third law gives the relationship between the orbital period, t days, of a planet around the Sun, and its **mean** distance, d km, from the Sun.

In simple terms, this law states that t^2 is directly proportional to d^3.

Find a formula relating t and d, given that the Earth is 150 million km from the Sun.

Copy and complete the table.

PLANET	d (MILLION KM)	ORBITAL PERIOD AROUND SUN t (Earth days)
Mercury	57.9	
Jupiter		4315

Find the values of d and t for other planets in the Solar System, and see if they fit the same relationship.

INVERSE PROPORTION

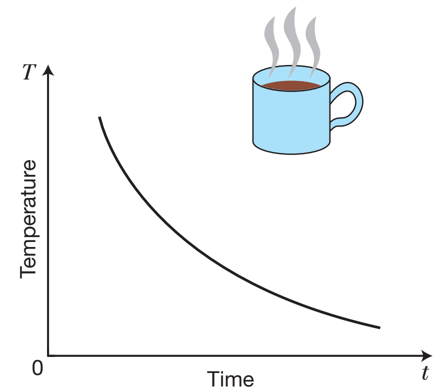
The temperature of a cup of coffee decreases as time increases.

A graph of temperature (T) against time (t) shows an **inverse** relationship.

This can be expressed as: 'T is inversely proportional to t'.

In symbols, this is written as $T \propto \frac{1}{t}$

The $\propto$ sign can then be replaced by '$= k$' to give the formula $T = \frac{k}{t}$ where k is the constant of proportionality.



EXAMPLE 4

SKILLS
ANALYSIS

Express these equations as relationships with constants of proportionality.

a y is inversely proportional to x squared.

b m varies inversely as the cube of n.

c s is inversely proportional to the square root of t.

d v squared varies inversely as the cube of w.

a $y \propto \frac{1}{x^2} \Rightarrow y = \frac{k}{x^2}$

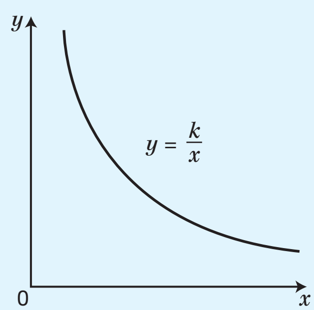
b $m \propto \frac{1}{n^3} \Rightarrow m = \frac{k}{n^3}$

c $s \propto \frac{1}{\sqrt{t}} \Rightarrow s = \frac{k}{\sqrt{t}}$

d $v^2 \propto \frac{1}{w^3} \Rightarrow v^2 = \frac{k}{w^3}$

KEY POINT

- If y is inversely proportional to x then $y \propto \frac{1}{x}$ and $y = \frac{k}{x}$, for some constant k. The graph of y plotted against x looks like this.



EXAMPLE 5

SKILLS
PROBLEM SOLVING

Sound intensity, I dB (decibels), is inversely proportional to the square of the distance, d m, from the source. At a music festival, it is 110 dB, 3 m away from a speaker.

a Find the formula relating I and d.

b Calculate the sound intensity 2 m away from the speaker.

c At what distance away from the speakers is the sound intensity 50 dB?

a I is inversely proportional to d^2 $\quad I = \frac{k}{d^2}$

$I = 110$ when $d = 3$ $\quad 110 = \frac{k}{3^2}$

$k = 990$

The formula is therefore $I = \frac{990}{d^2}$

b When $d = 2$ $\quad I = \frac{990}{2^2} = 247.5$

The sound intensity is 247.5 dB, 2 m away. (This is enough to cause deafness.)

c When $I = 50$ $\quad 50 = \frac{990}{d^2}$

$d^2 = 19.8$

$d = 4.45$ (3 s.f.)

The sound intensity is 50 dB, 4.45 m away from the speakers.

EXERCISE 3

1 ▶ y is inversely proportional to x. If $y = 4$ when $x = 3$, find

a the formula for y in terms of x
b y when $x = 2$
c x when $y = 3$
d Sketch the graph of y against x.

2 ▶ d varies inversely with t. If $d = 10$ when $t = 25$, find

a the formula for d in terms of t
b d when $t = 2$
c t when $d = 50$

3 ▶ The pressure, P (in N/m^2), of a gas is inversely proportional to the volume, V (in m^3). $P = 1500$ N/m^2 when $V = 2$ m^3

a Write a formula for P in terms of V.
b Work out the pressure when the volume of the gas is 1.5 m^3.
c Work out the volume of gas when the pressure is 1200 N/m^2.
d What happens to the volume of the gas when the pressure doubles?

4 ▶ The time taken, t (in seconds), to boil water in a kettle is inversely proportional to the power, p (in watts) of the kettle.
A full kettle of power 1500 W boils the water in 400 seconds.

a Write a formula for t in terms of p.
b A similar kettle has a power of 2500 W. Can this kettle boil the same amount of water in less than 3 minutes?

5 ▶ m varies inversely with the square of n. If $m = 4$ when $n = 3$, find

a the formula for m in terms of n
b m when $n = 2$
c n when $m = 1$

6 ▶ V varies inversely with the cube of w. If $V = 12.5$ when $w = 2$, find

a the formula for V in terms of w
b V when $w = 1$
c w when $V = 0.8$

7 ▶ Light intensity, I candle-power, is inversely proportional to the square of the distance, d m, of an object from this light source. If $I = 10^5$ when $d = 2$ m, find

a the formula for I in terms of d
b the light intensity at 2 km.

8 ▶ The life-expectancy, L days, of a cockroach varies inversely with the square of the density, d people/m^2, of the human population near its habitat. If $L = 100$ when $d = 0.05$, find

a the formula for L in terms of d
b the life-expectancy of a cockroach in an area where the human population density is 0.1 people/m^2.

EXERCISE 3*

10th

1 ▶ The cost of Mrs Janus's electricity bill, $\$C$, varies inversely with the average temperature, t °C, over the period of the bill. If the bill is \$200 when the temperature is 25 °C, find

a the formula expressing C in terms of t
b the bill when the temperature is 18 °C
c the temperature generating a bill of \$400.

2 ▶ As a balloon is blown up, the thickness of its walls, t (mm), decreases and its volume, V (cm^3), increases. V is inversely proportional to t. When V is 15 000 cm^3, t is 0.05 mm.

a Write a formula for V in terms of t.
b When the thickness of the wall of the balloon is 0.03 mm, the balloon will pop. Is it possible to blow up this balloon to a volume of 30 000 cm^3?

3 ▶ y is inversely proportional to x.

x	0.25	0.5	1	2	4	5	10	20
y	48	24	12	6	3	2.4	1.2	0.6

a Draw a graph of y against x.
What type of graph is this?

b $y = \frac{k}{x}$ where k is the constant of proportionality. Find k.

c Work out $x \times y$ for each pair of values in the table. What do you notice?

4 ▶ a is inversely proportional to b^2. Copy and complete this table.

b	2	5	
a	50		2